Arius®
Interpolation

Milliman

IT TAKES VISION

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1. Interpolation

You may occasionally evaluate your data as of some date other than the end of a full exposure period. For example, you might have triangles of annual x annual data but the most recent evaluation is only one quarter rather than a full year. In this case, selected development factors based on the historical annual data should be adjusted to be more appropriate for use with the most recent interim diagonal in estimating ultimates.

Arius' interpolation capabilities can help by:

- Fitting a smooth curve between your known selected factors, for example selections based on data at 12-24-36-48, and then

- Using the resulting formulas to calculate new factors that are adjusted to more reasonably apply to a diagonal of data as of 3-15-27-39.

- The process of interpolating factors in Arius affects exhibits, methods, and some summary reports referencing the selected ultimate.

EXHIBITS

When working with full-period data (for example, an annual development period evaluated at 12 months), you typically use several different statistics on your development exhibits to help you select development factors. With interim period data, you can take this process a step further on those same exhibits. Arius provides several sets of interpolated factors, using various curve fits of your standard selected development factors, to help you select interpolated development factors. You can select which curves to calculate and display on development exhibits in EXHIBIT OPTIONS|INTERPOLATED FACTORS.

METHODS

Arius system methods referencing selected development factors contain a function **IfInterpolate** that identifies when the file structure includes a partial period. If the function is true (the last diagonal is a partial period based on the rest of the triangle), those methods will automatically refer to selected interpolated factors rather than standard selected development factors. You do not need to "turn on" interpolation. File structure can be viewed and modified in PROJECT SETTINGS|FILE STRUCTURE.
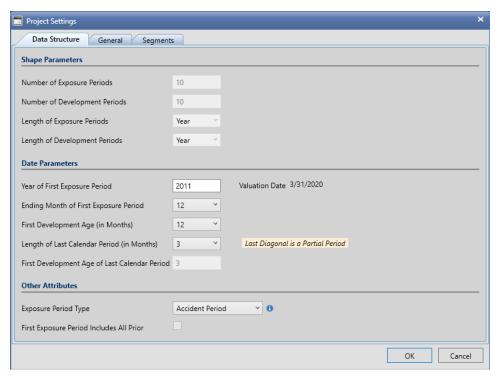
REPORTS

Arius provides several summary reports referencing selected ultimate amounts and counts. If the last diagonal of the file is a partial period, those ultimates will be prorated based on a selected proration amount so the ratios referencing ultimates will correctly calculate based on the prorated amount rather than the full amount.
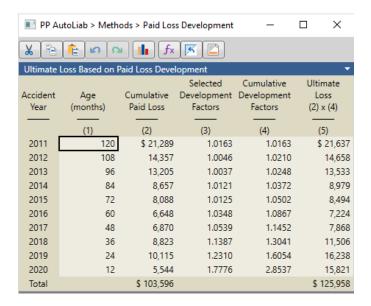
2. Using the Arius interpolation capabilities

Development factor interpolation and selected ultimate proration is activated in Arius based on file structure. You control which curve fits are displayed on development exhibits and make final selections, and those selections are then automatically used in all actuarial methods referencing development factors. Final ultimate selections will then be prorated using a selected proportion earned.

PREPARING YOUR PROJECT FOR INTERPOLATION

- If you are performing your first evaluation in a new development period, append a new diagonal to your Arius project using MODIFY STRUCTURE|APPEND NEW EVALUATION PERIOD.

- In PROJECT SETTINGS|DATA STRUCTURE|LENGTH OF LAST CALENDAR PERIOD (IN MONTHS) enter the number of months included in the latest evaluation period. When the number of months entered here indicates that this evaluation is shorter than the full development period, you will see the note **Last Diagonal is a Partial Period** reminding you that interpolation capabilities have been activated for this Arius project. Verify your entry by reviewing the calculated valuation date displayed on this screen.

- In the example below, the latest evaluation period is through the first quarter in a yearly analysis project.

DEFINING WHICH INTERPOLATED RESULTS DISPLAY ON EXHIBITS

EXHIBIT OPTIONS | INTERPOLATED FACTORS

- Use the buttons at the top of the window to add new interpolated results, delete existing ones from the list, and rearrange the list as desired.

- Click the drop-down button within a row to modify the curve and/or source of the calculation. Define your calculations in EXHIBIT OPTIONS|INTERPOLATED FACTORS.

  - **Curve Fit** – This drop-down list selector offers the following four types of curves to use for the interpolation calculations, each having the greatest impact on the least mature ages.

    Exponential
    Inverse Power
    Linear
    Weibull

  - **Input** - This drop-down list selector identifies the known_y values to be used as input which are based on your selected development factors. You have two options:
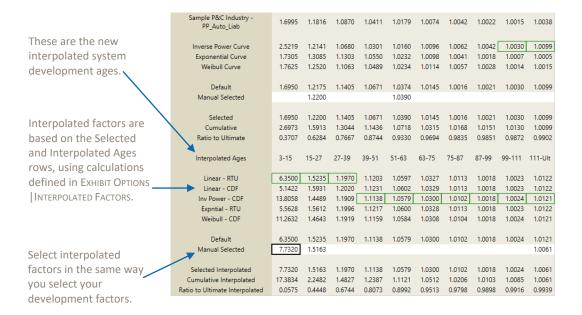
    Cumulative
    Ratio to ultimate values

  - **Abbreviate Labels** – Check this box if you want to display abbreviated labels to the left of your interpolation calculations on your exhibits. Otherwise longer-form row labels will be used.

  - Click **OK** to save your preferences and all development exhibits will now display those preferences (when PROJECT SETTINGS|LENGTH OF LAST CALENDAR PERIOD (MONTHS) is set to a partial period, thus activating interpolation in your Arius project).

> Note
>
> The calculated development factors that are displayed on exhibits are always *incremental* factors. The label to the left of these factors describe the curve fit and input data chosen in EXHIBIT OPTIONS |INTERPOLATED FACTORS for the interpolation calculation.

Development Exhibit with interpolation analysis

These are the new interpolated system development ages.

Interpolated factors are based on the Selected and Interpolated Ages rows, using calculations defined in EXHIBIT OPTIONS |INTERPOLATED FACTORS.

Select interpolated factors in the same way you select your development factors.

Sample P&C Industry - PP_Auto_Liab	1.6995	1.1816	1.0870	1.0411	1.0179	1.0074	1.0042	1.0022	1.0015	1.0038
Inverse Power Curve	2.5219	1.2141	1.0680	1.0301	1.0160	1.0096	1.0062	1.0042	1.0030	1.0099
Exponential Curve	1.7305	1.3085	1.1303	1.0550	1.0232	1.0098	1.0041	1.0018	1.0007	1.0005
Weibull Curve	1.7625	1.2520	1.1063	1.0489	1.0234	1.0114	1.0057	1.0028	1.0014	1.0015
Default	1.6950	1.2175	1.1405	1.0671	1.0374	1.0145	1.0016	1.0021	1.0030	1.0099
Manual Selected		1.2200			1.0390					
Selected	1.6950	1.2200	1.1405	1.0671	1.0390	1.0145	1.0016	1.0021	1.0030	1.0099
Cumulative	2.6973	1.5913	1.3044	1.1436	1.0718	1.0315	1.0168	1.0151	1.0130	1.0099
Ratio to Ultimate	0.3707	0.6284	0.7667	0.8744	0.9330	0.9694	0.9835	0.9851	0.9872	0.9902
Interpolated Ages	3-15	15-27	27-39	39-51	51-63	63-75	75-87	87-99	99-111	111-Ult
Linear - RTU	6.3500	1.5235	1.1970	1.1203	1.0597	1.0327	1.0113	1.0018	1.0023	1.0122
Linear - CDF	5.1422	1.5931	1.2020	1.1231	1.0602	1.0329	1.0113	1.0018	1.0023	1.0122
Inv Power - CDF	13.8058	1.4489	1.1909	1.1138	1.0579	1.0300	1.0102	1.0018	1.0024	1.0121
Expntial - RTU	5.5628	1.5612	1.1996	1.1217	1.0600	1.0328	1.0113	1.0018	1.0023	1.0122
Weibull - CDF	11.2632	1.4643	1.1919	1.1159	1.0584	1.0308	1.0104	1.0018	1.0024	1.0121
Default	6.3500	1.5235	1.1970	1.1138	1.0579	1.0300	1.0102	1.0018	1.0024	1.0121
Manual Selected	7.7320	1.5163								1.0061
Selected Interpolated	7.7320	1.5163	1.1970	1.1138	1.0579	1.0300	1.0102	1.0018	1.0024	1.0061
Cumulative Interpolated	17.3834	2.2482	1.4827	1.2387	1.1121	1.0512	1.0206	1.0103	1.0085	1.0061
Ratio to Ultimate Interpolated	0.0575	0.4448	0.6744	0.8073	0.8992	0.9513	0.9798	0.9898	0.9916	0.9939

SELECT INTERPOLATED DEVELOPMENT FACTORS

Similar to the standard development factor selections above, you can set defaults and/or manually select interpolated development factors. All of these options will be at the bottom of the development exhibit window.

- Set default interpolated development factor selections by right-clicking on a cell, group of cells, or row label and choosing DEFAULT|SET AS DEFAULT.

- When applying additional actuarial judgment, to override a default selection, manually enter your selection in the **Manual Selected** row (or right-click and select **Copy Value to Manual Selected**).

- Once interpolated development factor selections are made, all the methods in Arius referencing development factors will automatically reference these selected interpolated development factors.

3. Details behind the calculations

When you use loss development factors to project ultimate losses at the end of a full exposure period, you typically estimate the percentage paid or incurred for a series of development ages (x_i's) dictated by the evaluations of the data triangle. At quarter-end however, the latest diagonal is often an interim or partial period. Arius helps account for this by deriving an adjusted percentage paid or incurred for a new series of development ages by "drawing" a smooth curve through (and beyond) each x_i.

For example:

- Suppose your data triangles were evaluated as of year-end, and the associated development ages for your SDFs are x = 12-24-36…84.

- Next suppose your latest diagonal is an interim period as of March 31, with your new interpolated ages being x = 3-15-27…75.

- An interpolation technique can be used to convert your selected development pattern (known_y's) at 12-24-36…84 (known_x's) to a new interpolated development pattern (new_y's) at 3-15-27…75 (new_x's).

- You can then apply this new interpolated pattern to your latest diagonal of data to project an ultimate loss.

The interpolation formula used in Arius will return an array of new_y's for a series of new_x's by using existing known_x's and known_y's. The notation is as follows:

InterpolateFunction (known_y's, known_x's, new_x's)

- **Known_y's** represent the array of y-values you already know, e.g., your selected development pattern {.256 .541 .730 .806 .871 .900 .931}. This pattern can be described in terms of age-to-ultimate (cumulative) factors or percent of ultimate (ratio to ultimate) values.

- **Known_x's** represent the array of x-values you already know, e.g., the development ages associated with the known_y's above {12-24-36…84}. This array can be explicitly determined by the structure of your columned data (i.e. number of development periods, length of development periods, and starting period in months).

- **New_x's** represents the new array of x-values for which you want InterpolateFunction to return corresponding y-values, e.g., the development ages corresponding to the actual evaluation date of your latest diagonal {3-15-27…75}.

- **Function** relates to the equation used to fit the two known data points provided. Arius provides functions for the Exponential, Inverse Power, Linear, and Weibull curves. After reviewing your own data, you should find that at least one of the formulas provides a curve that resembles the tendencies in your data. See Interpolation Functions (below) for more detail.

Interpolation functions

Arius uses Exponential, Inverse Power, Linear, and Weibull functions for interpolation. Translated for purposes of running a linear regression and solving directly for the a and b parameters, the formulas in Arius are as follows (where y = actual loss development factor, x = age of development in months).

Exponential: $\quad y = e^{(a+bx)}$ $\qquad$ for y > 0

Inverse Power: $\quad y = \dfrac{1}{1 + e^a x^{-b}}$ $\qquad$ for 0 < y < 1

$\qquad\qquad\qquad\; y = 1 + e^a x^{-b}$ $\qquad$ for y > 1

Linear: $\qquad\quad\; y = a + bx$ $\qquad\qquad$ for y > 0

Weibull: $\qquad\; y = 1 - e^{-e^a x^b}$ $\qquad$ for 0 < y < 1

$\qquad\qquad\qquad\; y = \dfrac{1}{1 - e^{-e^a x^b}}$ $\qquad$ for y > 1

Given two known data points, the first step is to solve for the slope (b) and intercept (a). Thus we need to transform the above equations into a linear form, as follows:

Exponential: $\qquad \ln(y) = a + b$ $\qquad\qquad\qquad$ for y > 0

Inverse Power: $\qquad \ln(1/y - 1) = a + \ln(1/x)b$ $\qquad$ for 0 < y < 1

$\qquad\qquad\qquad\quad \ln(y - 1) = a + \ln - (1/x)b$ $\qquad$ for y > 1

Linear: $\qquad\qquad\; y = a + bx$ $\qquad\qquad\qquad\qquad$ for y > 0

Weibull: $\qquad\quad \ln\big(-\ln(1 - y)\big) = a + \ln(x)b$ $\qquad$ for 0 < y < 1

$\qquad\qquad\qquad\quad \ln\big(-\ln(1 - 1y)\big) = a + \ln(x)b$ $\qquad$ for y > 1

Once a and b are derived, the values are plugged back into the original formulas along with the new_x to arrive at the interpolated value or new_y.

Interpolation method

Arius uses the Average Date of Loss Approximation method in its interpolation. This method was first seen in actuarial literature in the article *Extrapolating, Smoothing, and Interpolating Development Factors* by Sherman, and later in *Analysis of Loss Development Patterns Using Infinitely Decomposable Percent of Ultimate Curves* by Robbin and Homer.

The key to this approach is in the handling of partial periods, e.g., the less than 12 month factor for an accident year period or less than 24 month factor for a policy year period. The Average Date of Loss Approximation formula takes into account two adjustments. The first adjusts for the maturity of a partial period and the second adjustment accounts for the unearned exposure of the partial period.

Maturity adjustment

A partial period evaluated at 3 months cannot be assumed to have the same maturity as a full period evaluated at 3 months. For example, an equivalent maturity for a 3-month partial period as of the end of the 3 months is equivalent to around a 7.5-month maturity for a full year period. Therefore, the development factor to ultimate for a 3-month period should be the same as the 7.5-month to ultimate factor for a 12-month period.

Another way to look at this is to calculate the difference between the evaluation date and the average accident date for each development factor you have, defined by M(x).

For an accident period:

$$M(x) = MAX(\frac{x}{2}, x - \frac{k_{exp}}{2}) \qquad \text{where} \qquad k_{exp} = \begin{cases} 12 \text{ if years} \\ 6 \text{ if half years} \\ 3 \text{ if quarters} \\ 1 \text{ if months} \end{cases}$$

For a policy period:

$$M(x) = \begin{cases} MAX(\frac{x}{3}, x - \frac{2k_{exp}}{3}) & \text{for x<=k exp} \\[2ex] \dfrac{(x - k_{exp}) + (\frac{1}{3})(2k_{exp} - x)(1 - G(x))}{G(x)} & \text{for x>k exp} \end{cases}$$

where G(x) represents the exposure adjustment described below.

For example, suppose you have a 3-month accident period evaluated at the end of 3 months. The difference between the evaluation date and the average accident date, M(x), would be 1.5 months. This would be equivalent to a 12-month accident period with an average accident date at 6 months, evaluated at 7.5 months.

Exposure adjustment

The second adjustment adjusts for the unearned exposure of the partial periods. For example, your 3-month to ultimate factor could project a full accident year ultimate or a partial accident year ultimate. If you want to project a full year ultimate, you will need to multiply your 3-month to ultimate factor times by a factor of 12/3 or 4 to account for the remaining 9 months of exposure. Or another way to look at it is to divide your resulting interpolated age-to-ultimate factor by your cumulative exposure, defined as G(x).

For an accident period:

$$G(x) = MIN(\frac{x}{k_{exp}},1) \qquad \text{where} \qquad k_{exp} = \begin{cases} 12 \text{ if years} \\ 6 \text{ if half years} \\ 3 \text{ if quarters} \\ 1 \text{ if months} \end{cases}$$

For a policy period:

$$G(x) = \begin{cases} 0.5 * MIN(\frac{x}{k_{exp}},1)^2 & \text{for } x <= k \text{ exp} \\ 1 - 0.5 * MAX(2 - \frac{x}{k_{exp}},0)^2 & \text{for } x > k \text{ exp} \end{cases}$$

Sample interpolation procedure

The easiest way to walk through the steps of the procedure is to provide an example.

Suppose you had the following selected development pattern at 12-24-36…84 months and you need to interpolate the factors to 3-15-27…75 months. Also assume your exposure period represents an accident year (k_{exp} = 12).

Age	12-24	24-36	36-48	48-60	60-72	72-84	84-Ult
Selected	2.110	1.350	1.105	1.080	1.034	1.034	1.074
Cumulative	3.903	1.850	1.370	1.240	1.148	1.111	1.074
Ratio to Ultimate	0.256	0.541	0.730	0.806	0.871	0.900	0.931

Step 1: Identify array of known_x's, adjusted for Maturity

For an array of known_x's = {12-24-36-48-60-72-84} and a given exposure period length k_{exp} = 12 months, the maturity adjustment M(x) calculates the difference between the evaluation date and average accident date for each development factor that you have.

$$M(x) = MAX(\frac{x}{2}, x - \frac{k_{exp}}{2})$$ so adjusted known_x's = {6-18-30-42-54-66-78}

Step 2: Identify array of known_y's

The known_y's can be chosen from among the cumulative development factors or ratio to ultimate factors.

Using Ratio to Ultimates, known_y's = {.256 .541 .730 .806 .871 .900 .931}

Step 3: Identify array of new_x's, adjusted for Maturity

For an array of new_x's = {3-15-27-39-51-63-75} and a given exposure period length k_{exp} = 12 months, the maturity adjustment M(x) calculates the difference between the evaluation date and the average accident date for each new development factor you want to calculate.

$$M(x) = MAX(\frac{x}{2}, x - \frac{k_{exp}}{2})$$ so adjusted new_x's = {1.5-9-21-33-45-57-69}

Step 4: Find the 2 closest known_y's

For each adjusted new_x, find the two closest known data points that border each side of the development factors you want. If you are extrapolating past the last development factor, use the last two factors, and if you are extrapolating to ages earlier than you have data, use the first two factors. You will use these factors for the interpolation.

new_x1 = 1.5   so two closest data points =   {6,0.256} and {18,0.541}

new_x2 = 9   so two closest data points =   {6,0.256} and {18,0.541}

new_x3 = 21   so two closest data points =   {18,0.541} and {30,0.730}

etc.

Step 5: Adjust the known_y's for cumulative exposure

The next step is to adjust the development factors that you have for the amount of cumulative exposure. To accomplish this, multiply each of the factors by their cumulative exposure G(x) if the factors you are using are cumulative development factors; or divide each of the factors by their cumulative exposure G(x) if the factors you are using are ratio to ultimate factors, where x = known_x's.

Since the factors in this example begin at 12 months, the cumulative exposure

$$G(x) = MIN(\frac{x}{12},1)$$ = 1, so none of the known_y's need to be adjusted.

Step 6: Apply an interpolation formula to calculate interpolated factor

Arius allows you to choose among several interpolation functions including Exponential, Inverse Power, Linear, and Weibull. The approach to arrive at the new interpolated factors is the same. The first step is to calculate an intercept (a) and slope (b) given the two adjusted data points using the transformed linear equations for each function. The next step is to plug the parameters back into the original equations to arrive at a new_y.

Exponential: Given $x1 = 6$ and $\text{In}(y1) = -1.363$

    and $x2 = 18$ and $\text{In}(y2) = -0.164$

    $b = 0.062$

    $a = -1.735$

So y = $e^{a+bx} = e^{-1.735+0.062*1.5} = 0.194$

Inverse Power: Given $\text{In}(1/x_1) = -1.792$ and $\text{In}(1/y_1\text{-}1) = 1.066$

    and $\text{In}(1/x_2) = -2.890$ and $\text{In}(1/y_2\text{-}1) = 0.163$

    $b = 1.118$

    $a = 3.069$

So y = $1/(1 + e^{a}x^{-b}) = 1/(1+e^{3.069*1.5^{-1.118}}) = 0.068$

Linear:	Given	$x_1 = 6$ and $y_1 = 0.256$
	and	$x_2 = 18$ and $y_2 = 0.541$
		$b = 0.024$
		$a = 0.114$
		$y = a + bx = 0.114 + 0.024*1.5 = 0.150$

Weibull:	Given	$\text{In}(x1) = 1.792$ and $\text{In}\big(\text{-In}(1 - y1)\big) = -1.218$
	and	$\text{In}(x2) = 2.890$ and $\text{In}\big(-\text{In}(1 - y2)\big) = -0.250$
		$b = 0.879$
		$a = -2.793$
	So y =	$1 - e\hat{\ }\big((-e^a)x^b\big) =$
		$1 - e\hat{\ }\big((-e^{-2.793})*1.5^{0.879}\big) = 0.083$

Step 7: Adjust the new_y's for cumulative exposure

Up to this point, the interpolated factor from age x-to-ultimate is only projecting the partial 3-month period ultimate for this example. The final step therefore is to adjust the development factors for the amount of cumulative exposure. To accomplish this, divide each of the factors by their cumulative exposure G(x) if the factors you are using are selected or cumulative development factors; or multiply each of the factors by their cumulative exposure G(x) if the factors you are using are ratio to ultimate factors, where x = new_x's.

For this example, $\qquad G(x) = MIN(\dfrac{x}{12}, 1) = \quad MIN(\dfrac{3}{12}, 1) \quad = 0.250$

So the adjusted new_y's would be calculated as follows:

Exponential:	y = 0.194 * 0.250 = 0.049
Inverse Power:	y = 0.068 * 0.250 = 0.017
Linear:	y = 0.150 * 0.250 = 0.038
Weibull:	y = 0.083 * 0.250 = 0.021

4. Quick reference

Define the project structure and preferred interpolated results

1. If required, append a new diagonal to your Arius project by selecting MODIFY STRUCTURE|APPEND NEW EVALUATION PERIOD.

2. In PROJECT SETTINGS|DATA STRUCTURE, enter the **Length of Last Calendar Period (in months)**.

3. Open EXHIBIT OPTIONS|INTERPOLATED FACTORS and select your preferred curves and inputs for interpolated factors.

4. Load your partial period input data into the last diagonal.

Select interpolated development factors

1. Follow the steps above for displaying interpolation calculations on exhibits.

2. Open a development exhibit and select development factors (or confirm existing full period selections).

3. In the interpolation section at the bottom of the exhibit, either set defaults in the interpolated factor rows (right-click on a factor, set of factors, or row label then select DEFAULT|SET AS DEFAULT), or enter selections in the **Manual Selected** row.

Reference interpolated factors in user-defined tables

- To reference selected interpolated development factors in a formula, select **Interpolated Development Factors** from the **Source Array** category drop-down list in the **Formula Editor** dialog box.

- To calculate interpolated factors outside of the development exhibits, for example directly in a method or report, use the interpolation functions. You will find these functions in the Formula Editor dialog box under the All or Mathematical category. Interpolation functions include:

    InterpolateExponential
    InterpolateInversePower
    InterpolateLinear
    InterpolateWeibull

- Use the **IfInterpolate** function in user-defined tables to allow the file structure to manage when interpolated factors are used in the table.